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Constant of Motion, Lagrangian and Hamiltonian of the Gravitational Attraction of Two Bodies with Variable Mass

G. López[1]

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The Lagrangian, the Hamiltonian and the constant of motion of the gravitational attraction of two bodies when one of them has variable mass is considered. The relative and center of mass coordinates are not separated, and choosing the reference system in the body with much higher mass, it is possible to reduce the system of equations to 1-D problem. Then, a constant of motion, the Lagrangian, and the Hamiltonian are obtained. The trajectories found in the space position-velocity,(x, v), are qualitatively different from those on the space position-momentum,(x, p).

KEY WORDS: constant of motion; Lagrangian; Hamiltonian; variable mass.

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1. INTRODUCTION

Mass variable systems has been important since the foundation of the classical mechanics and have been relevant in modern physics (López *et al.*, 2004). Among these type of systems one could mentioned: the motion of rockets (Sommerfeld, 1964), the kinetic theory of dusty plasmas (Zagorodny *et al.*, 2000), propagation of electromagnetic waves in dispersive and nonlinear media (Serimaa *et al.*, 1986), neutrinos mass oscillations (Bethem, 1986; Commins and Bucksbaum, 1983), black holes formation (Helhi *et al.*, 1998), and comets interacting with solar wind (Biermann, 1971; Nuth *et al.*, 2000; Reeves, 1974). The interest in this last system comes from the concern about to determinate correctly the trajectory of the comet as its mass is changing. This system belong to the so called two-bodies problem. The gravitational two-bodies system is one of the must well known systems in classical mechanics (Goldstein, 1950) and is the system which made a revolution in our planetary and cosmological concepts. Normally, one assumes that the masses

[1] Departamento de Física de la Universidad de Guadalajara, Apartado Postal 4-137, 44410 Guadalajara, Jalisco, México; e-mail: gulopez@udgserv.cencar.udg.mx

of these two bodies are fixed and unchanged during the dynamical interaction (Heisler, 1986; Matese *et al.*, 1991). However, this can not be true any more when one consider comets as one of the bodies. Comets loose part of their mass as traveling around the sun (or other star) due to their interaction with the solar wind which blows off particles from their surfaces. In fact, it is possible that the comet could disappear as it approaches to the sun (Hughes, 1984; Stern and Weissman, 2001). So, one should consider the problem of having one body with variable mass during its gravitational interaction with other body.

In this paper, one considers the problem of finding the constant of motion, Lagrangian, and Hamiltonian, for the gravitational interaction of two bodies when one of them is loosing its mass during the gravitational interaction. The mass of one of the bodies is assumed much larger than the mass of the other body. Choosing the reference system on big-mass body, the three-dimensional two-bodies problem is reduced to a one-dimensional problem. Then, one uses the constant of motion approach (Kobussen, 1979; Leubner, 1981; López, 1996) to find the Lagrangian and the Hamiltonian of the system. A model for the mass variation is given for an explicit illustration of form of these quantities. With this model, one shows that the trajectories in the space position-velocity (defined by the constant of motion) are different than the trajectories on the space position-momentum (defined by the Hamiltonian).

2. REFERENCE SYSTEM AND CONSTANT OF MOTION

Newton's equations of motion for two bodies interacting gravitationally, seen from arbitrary inertial reference system, are given by

$$\frac{d}{dt}\left(m_1\frac{d\mathbf{r_1}}{dt}\right) = -\frac{Gm_1m_2}{|\mathbf{r_1}-\mathbf{r_2}|^3}(\mathbf{r_1}-\mathbf{r_2}) \tag{1a}$$

and

$$\frac{d}{dt}\left(m_2\frac{d\mathbf{r_2}}{dt}\right) = -\frac{Gm_1m_2}{|\mathbf{r_2}-\mathbf{r_1}|^3}(\mathbf{r_2}-\mathbf{r_1})\ , \tag{1b}$$

where m_1 and m_2 are the masses of the bodies, $\mathbf{r_1} = (x_1, y_1, z_1)$ and $\mathbf{r_2} = (x_2, y_2, z_2)$ are the vectors position of the two bodies from our reference system, G is the gravitational constant, and

$$|\mathbf{r_1}-\mathbf{r_2}| = |\mathbf{r_2}-\mathbf{r_1}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

is the Euclidean distance between the two bodies. It will be assumed that m_1 is constant and that m_2 varies with respect the time. Taking into consideration this mass variation, Eqs. (1a) and (1b) are written as

$$m_1\frac{d^2\mathbf{r_1}}{dt^2} = -\frac{Gm_1m_2}{|\mathbf{r_1}-\mathbf{r_2}|^3}(\mathbf{r_1}-\mathbf{r_2}) \tag{2}$$

and

$$m_2 \frac{d^2\mathbf{r_2}}{dt^2} = -\frac{Gm_1m_2}{|\mathbf{r_2}-\mathbf{r_1}|^3}(\mathbf{r_2}-\mathbf{r_1}) - \dot{m}_2\frac{d\mathbf{r_2}}{dt} \,, \tag{3}$$

where it has been defined $\dot{m}_2$ as $\dot{m}_2 = dm_2/dt$. Now, let us consider the usual relative, $\mathbf{r}$, and center of mass, $\mathbf{R}$, coordinates defined as

$$\mathbf{r} = \mathbf{r_2} - \mathbf{r_1} \,, \qquad \text{and} \qquad \mathbf{R} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1+m_2} \,. \tag{4}$$

Let us then differentiate twice these coordinates with respect the time, taking into consideration the equations (2) and (3). Thus, the following equations are obtained

$$\ddot{\mathbf{r}} = -\frac{(m_1+m_2)G}{r^3}\,\mathbf{r} - \frac{\dot{m}_2}{m_2}\,\dot{\mathbf{r}}_2 \tag{5}$$

and

$$\ddot{\mathbf{R}} = \frac{-\dot{m}_2}{m_1+m_2}\,\dot{\mathbf{r}}_2 + \frac{2m_2\dot{m}_2}{(m_1+m_2)^2}\,\dot{\mathbf{r}} + \frac{(m_1+m_2)m_1\ddot{m}_2 - 2m_1\dot{m}_2^2}{(m_1+m_2)^3}\,\mathbf{r} \,. \tag{6}$$

One sees that the relative motion does not decouple from the center of mass motion. So, these new coordinates are not really useful to deal with mass variation systems. In fact, using (4) , one has

$$\mathbf{r_2} = \mathbf{R} + \frac{m_1}{m_1+m_2}\,\mathbf{r} \,, \quad \text{and} \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} + \frac{m_1}{m_1+m_2}\dot{\mathbf{r}} - \frac{m_1\dot{m}_2}{(m_1+m_2)^2}\mathbf{r} \,. \tag{7}$$

Substituting these expressions in (5) and (6), one can see more clearly this coupling,

$$\ddot{\mathbf{r}} = \left[\frac{m_1+m_2)G}{r^3} + \frac{m_1\dot{m}_2^2}{m_2(m_1+m_2)^2}\right]\mathbf{r} - \frac{\dot{m}_2}{m_2}\left[\dot{\mathbf{R}} + \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right] \tag{8}$$

and

$$\ddot{\mathbf{R}} = \frac{-\dot{m}_2}{m_1+m_2}\mathbf{R} + \frac{m_1\dot{m}_2}{(m_1+m_2)^2}\dot{\mathbf{r}} + \frac{(m_1+m_2)m_1\ddot{m}_2 - m_1\dot{m}_2^2}{(m_1+m_2)^3}\,\mathbf{r} \,. \tag{9}$$

However, one can consider the case for $m_1 \gg m_2$ (which is the case star-comet), and consider to put our reference system just on the first body ($\mathbf{r_1} = \vec{\mathbf{0}}$). In this case, Eq. (3) becomes

$$m_2\frac{d^2\mathbf{r}}{dt^2} = -\frac{Gm_1m_2}{r^3}\,\mathbf{r} - \dot{m}_2\dot{\mathbf{r}} \,, \tag{10}$$

where $\mathbf{r} = \mathbf{r_2} = (x, y, z)$. Using spherical coordinates (r, θ, φ),

$$x = r\sin\theta\cos\varphi \,, \quad y = r\sin\theta\sin\varphi \,, \quad z = r\cos\theta \,, \tag{11}$$

Equation (10) can be written as

$$m_2\frac{d^2\mathbf{r}}{dt^2} = -\left[\frac{Gm_1m_2}{r^2} + \dot{m}_2\dot{r}\right]\widehat{r} + \dot{m}_2\left(r\dot{\theta}\,\widehat{\theta} + r\dot{\varphi}\sin\theta\,\widehat{\varphi}\right), \tag{12}$$

where $\widehat{r}$, $\widehat{\theta}$ and $\widehat{\varphi}$ are unitary directional vectors,

$$\widehat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, \quad \text{with} \quad \dot{\widehat{r}} = \dot{\theta}\,\widehat{\theta} + \dot{\varphi}\sin\theta\,\widehat{\varphi} \tag{13a}$$

$$\widehat{\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \quad \text{with} \quad \dot{\widehat{\theta}} = -\dot{\theta}\,\widehat{r} + \dot{\varphi}\cos\theta\,\widehat{\varphi} \tag{13b}$$

and

$$\widehat{\varphi} = (-\sin\theta, \cos\theta, 0)\,, \qquad \text{with} \qquad \dot{\widehat{\varphi}} = \sin\theta\,\widehat{r} + \cos\theta\,\widehat{\theta}\,. \tag{13c}$$

Since one has that $\mathbf{r} = r\widehat{r}$, it follows that

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 + r\dot{\varphi}\sin^2\varphi)\widehat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} + r\dot{\varphi}\sin\theta\cos\theta)\widehat{\theta} \\ &\quad + (2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta)\widehat{\varphi}\end{aligned} \tag{14}$$

and Eq. (12) is discomposed in the following three equations

$$m_2(\ddot{r} - r\dot{\theta}^2 + r\dot{\varphi}\sin^2\varphi) = -\frac{Gm_1m_2}{r^2} - \dot{m}_2\dot{r} \tag{15a}$$

$$m_2(2\dot{r}\dot{\theta} + r\ddot{\theta} + r\dot{\varphi}\sin\theta\cos\theta) = \dot{m}_2 r\dot{\theta}\,, \tag{15b}$$

and

$$m_2(2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta) = \dot{m}_2 r\dot{\varphi}\sin\theta\,. \tag{15c}$$

Thus, one has obtained coupling among these coordinates due to the term $\dot{m}_2$. Nevertheless, one can restrict oneself to consider the case $\dot{m}_2 r \approx 0$. For this case, it follows that $\dot{\varphi} = 0$, and the resulting equations are

$$m_2(\ddot{r} - r\dot{\theta}^2) = -\frac{Gm_1m_2}{r^2} - \dot{m}_2\dot{r}\,, \tag{16a}$$

and

$$m_2(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0\,. \tag{16b}$$

Let m_o be the mass of the second body when this one is very far away from the first body (when a comet is very far away from the sun, the mass of the comet remains constant). Since $m_2 \neq 0$ on (16b), the expression inside the parenthesis must be zero. In addition, one can multiply this expression by $m_o r$ to get the following constant of motion

$$P_\theta = m_o r^2\dot{\theta}\,. \tag{17}$$

Using this constant of motion in (16a), one obtains the equation

$$\frac{d^2r}{dt^2} = -\frac{Gm_1}{r^2} + \frac{P_\theta^2}{m_o^2 r^3} - \frac{\dot{m}_2}{m_2}\left(\frac{dr}{dt}\right) . \tag{18}$$

This equation represents a dissipative system for $\dot{m}_2 > 0$ and anti-dissipative system for $\dot{m}_2 < 0$. Suppose now that m_2 is a function of the distance between the first and second body, $m_2 = m_2(r)$. Therefore, it follows that

$$\frac{dm_2}{dt} = \frac{dm_2}{dr}\frac{dr}{dt} , \tag{19}$$

and Eq. (18) can be written as

$$\frac{d^2r}{dt^2} = -\frac{Gm_1}{r^2} + \frac{P_\theta^2}{m_o^2 r^3} - \frac{m_2'}{m_2}\left(\frac{dr}{dt}\right)^2 , \tag{20}$$

where $m_2' = dm_2/dr$. This equation can be seen as the following autonomous dynamical system (Drazin, 1992)

$$\frac{dr}{dt} = v , \qquad \frac{dv}{dt} = -\frac{Gm_1}{r^2} + \frac{P_\theta^2}{m_o^2 r^3} - \frac{m_2'}{m_2}v^2 . \tag{21}$$

A constant of motion for this system is a function $K = K(r, v)$ such that the following partial differential equation is satisfied (López, 1996)

$$v\frac{\partial K}{\partial r} + \left(\frac{-Gm_1}{r^2} + \frac{P_\theta^2}{m_o^2 r^3} - \frac{m_2'}{m_2}v^2\right)\frac{\partial K}{\partial v} = 0 . \tag{22}$$

This equation can be solved by the characteristic method (John, 1974) from which the following characteristic curve results

$$C(r, v) = m_2^2(r)v^2 + 2Gm_1\int\frac{m_2^2(r)\,dr}{r^2} - \frac{2P_\theta^2}{m_o^2}\int\frac{m_2^2(r)\,dr}{r^3} , \tag{23}$$

and the general solution of (22) is given by

$$K(r, v) = F(C(r, v)) , \tag{24}$$

where F is an arbitrary function of the characteristic curve. One can have a constant of motion with units of energy by selecting F as $F = C/2m_o$. That is, the constant of motion is given by

$$K(r, v) = \frac{m_2^2(r)}{2m_o}v^2 + \frac{Gm_1}{m_o}\int\frac{m_2^2(r)\,dr}{r^2} - \frac{P_\theta^2}{m_o^3}\int\frac{m_2^2(r)\,dr}{r^3} . \tag{25}$$

3. LAGRANGIAN AND HAMILTONIAN

Given the time independent constant of motion (25), the Lagrangian of the system (20) can be obtained using the following known expression (Kobussen, 1979; Leubner, 1981; López, 1996)

$$L(r, v) = v \int \frac{K(r, v)\, dv}{v^2} . \tag{26}$$

Thus, the Lagrangian is given by

$$L(r, v) = \frac{m_2^2(r)}{2m_o} v^2 - \frac{Gm_1}{m_o} \int \frac{m_2^2(r)\, dr}{r^2} + \frac{P_\theta^2}{m_o^3} \int \frac{m_2^2(r)\, dr}{r^3} . \tag{27}$$

The generalized linear momentum ($p = \partial L/\partial v$) is

$$p = \frac{m_2^2(r)}{m_o} v , \tag{28}$$

and the Hamiltonian is

$$H(r, p) = \frac{m_o p^2}{2m_2^2(r)} + \frac{Gm_1}{m_o} \int \frac{m_2^2(r)\, dr}{r^2} - \frac{P_\theta^2}{m_o^3} \int \frac{m_2^2(r)\, dr}{r^3} . \tag{29}$$

Note from (25) and (29) that the constant of motion and Hamiltonian can be written as

$$K(r, v) = \frac{m_2^2(r)}{2m_o} v^2 + V_{\text{eff}}(r) \tag{30}$$

and

$$H(r, p) = \frac{m_o p^2}{2m_2^2(r)} + V_{\text{eff}}(r) , \tag{31}$$

where V_{eff} is the effective potential energy defined as

$$V_{\text{eff}}(r) = \frac{Gm_1}{m_o} \int \frac{m_2^2(r)\, dr}{r^2} - \frac{P_\theta^2}{m_o^3} \int \frac{m_2^2(r)\, dr}{r^3} . \tag{32a}$$

This potential energy has an extreme value at the point

$$r^* = \frac{P_\theta^2}{Gm_1 m_o^2} \tag{32b}$$

which depends on m_o but it does not depend on the model for $m_2(r)$. One can see that this extreme value is a minimum for $m_2(r^*) \neq 0$, since one has that

$$\left(\frac{d^2 V_{\text{eff}}}{dr^2}\right)_{r=r^*} = \frac{(Gm_1 m_o)^4 m_o m_2^2(r^*)}{P_\theta^6} > 0 . \tag{33}$$

On the other hand, because of the expression (28), one could expect different behavior of a trajectory in the phase space (r, v) and the phase space (r, p). The trajectory $r(\theta)$ is found using the relation $dr/dt = (dr/d\theta)\dot{\theta}$, and the Eq. (17) in (30) to get

$$\int_{\theta_o}^{\theta} d\theta = \frac{P_\theta}{\sqrt{2m_o^3}} \int_{r_o}^{r} \frac{m_2(r)\, dr}{r^2\sqrt{K - V_{\text{eff}}(r)}} \,, \tag{34a}$$

where K and P_θ are determinate by the initial conditions, $K = K(r_o, v_o)$ and $P_\theta = m_o r_o^2 \dot{\theta}_o$. The time of half of cycle of oscillation, $T_{1/2}$, is obtained directly from Eq. (30) as

$$T_{1/2} = \frac{1}{\sqrt{2m_o}} \int_{r_1}^{r_2} \frac{m_2(r)\, dr}{\sqrt{K - V_{\text{eff}}(r)}} \,, \tag{34b}$$

where r_1 and r_2 are the two return points deduced as the solution of the following equation

$$V_{\text{eff}}(r_i) = K \,, \quad i = 1, 2 \,. \tag{34c}$$

4. MODEL OF VARIABLE MASS

As a possible application of (25) and (29), consider that a comet looses material as a result of the interaction with star wind in the following way (for one cycle of oscillation)

$$m_2(r) = \begin{cases} m_{oo}\sqrt{1 - e^{-\alpha r}} & \text{incoming } (v < 0) \\ m_i e^{\alpha(r_1 - r)} + m_f(1 - e^{-\alpha r}) & \text{outgoing } (v > 0) \end{cases} \tag{35}$$

where m_{oo} or m_f (where $m_f = 2m_i - m_{oo}$ by symmetry) is the mass of the comet very far away from the star (in each case), m_i is the mass of the comet at the closets approach to the star (distance r_1), $m_i = m_{oo}\sqrt{1 - e^{-\alpha r_1}}$, and α is a factor that can be adjusted from experimental data. Thus, the effective potential (32a) has the following form for the incoming case $(m_o = m_{oo})$

$$\begin{aligned} V_{\text{eff}}^{(\text{in})}(r) = &-\frac{Gm_1 m_{oo}}{r}(1 - e^{-\alpha r}) + \frac{P_\theta^2}{2m_{oo}r^2}(1 - e^{-\alpha r}) \\ &+ \left[GM_1 m_{oo}\alpha + \frac{\alpha^2 P_\theta^2}{2m_{oo}}\right] Ei(-\alpha r) + \frac{\alpha P_\theta^2 e^{-\alpha r}}{2m_{oo} r} \,, \end{aligned} \tag{36a}$$

where $Ei(x)$ is the exponential-integral function (Gradshteyn and Ryzhikm, 1980).

For the outgoing case, one has $m_o = m_f$ and

$$\begin{aligned} V_{\text{eff}}^{(\text{out})}(r) = &-\frac{Gm_1 m_f}{r} + \frac{\tilde{P}_\theta^2}{2m_f r^2} \\ &+ \frac{Gm_1(m_i e^{\alpha r_1} - m_f)^2}{m_f}\left[-\frac{e^{-2\alpha r}}{r} - 2\alpha Ei(-2\alpha r)\right] \end{aligned}$$

$$-\frac{\tilde{P}_\theta^2(m_i e^{\alpha r_1}-m_f)^2}{m_f^3}\left[-\frac{e^{-2\alpha r}}{2r^2}+\frac{\alpha e^{-2\alpha r}}{r}+2Ei(-2\alpha r)\right]$$
$$+2Gm_1(m_i e^{\alpha r_1}-m_f)\left[-\frac{e^{-\alpha r}}{r}-\alpha Ei(-\alpha r)\right]$$
$$-\frac{2\tilde{P}_\theta^2(m_i e^{\alpha r_1}-m_f)}{m_f^2}\left[-\frac{e^{-\alpha r}}{2r^2}+\frac{\alpha e^{-\alpha r}}{2r}+\frac{\alpha^2}{2}Ei(-\alpha r)\right] \quad (36b)$$

where $\tilde{P}_\theta$ is defined now as $\tilde{P}_\theta = m_f r^2\dot{\theta}$. The extreme point of the effective potential (32b) for the incoming and outgoing cases is given by

$$r^*_{\text{in}} = \frac{P_\theta^2}{Gm_1 m_{oo}^2}\,, \qquad r^*_{\text{out}} = \frac{P_\theta^2}{Gm_1 m_f^2} \quad (37)$$

Given the definition (35), the constant of motion, Lagrangian, generalized linear momentum, and Hamiltonian are given by

$$K^{(i)}(r,v) = \frac{m_2^2(r)}{2m_o}v^2 + V^{(i)}_{\text{eff}}(r), \quad (38)$$

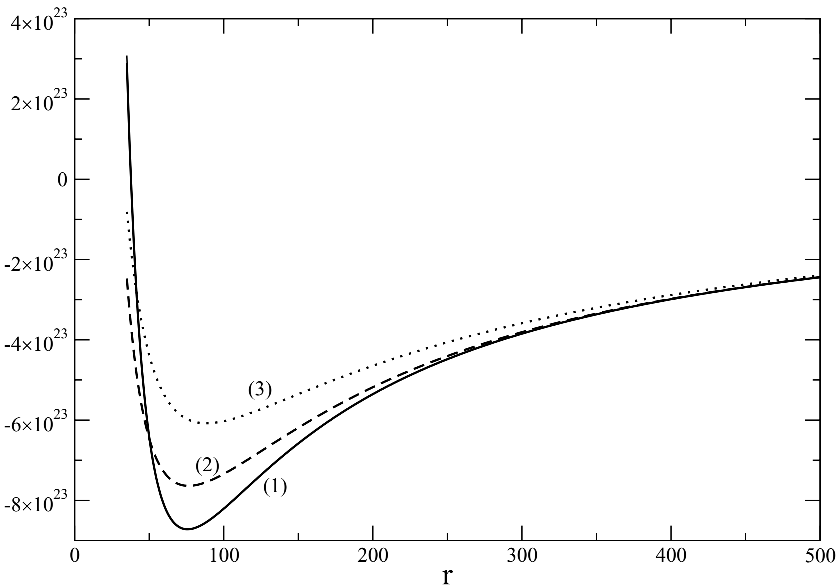

Fig. 1. $V^{(\text{in})}_{\text{eff}}(r)$ with the values of the parameters given on (42), for $\alpha = 1$ (1); $\alpha = 0.01$ (2); and $\alpha = 0.005$ (3).

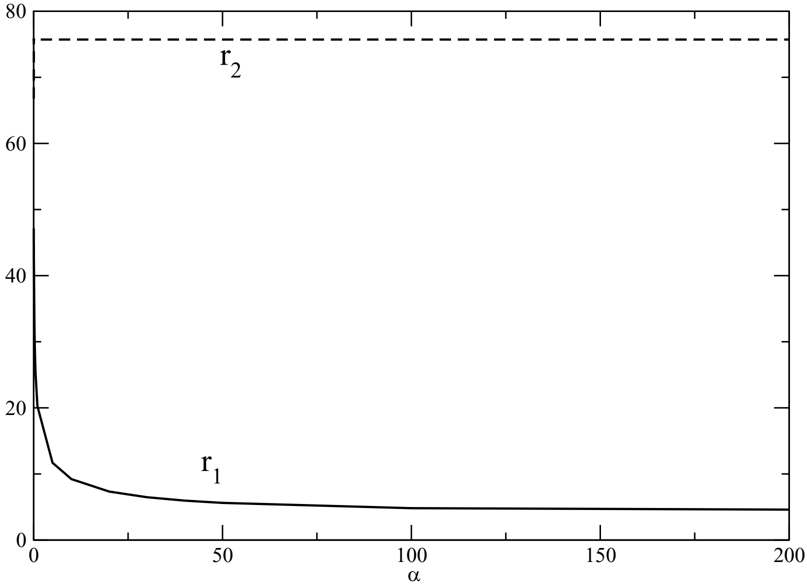

Fig. 2. Maximum (r_2) and minimum (r_1) distances between the two bodies as a function of the parameter α.

$$L^{(i)}(r, v) = \frac{m_2^2(r)}{2m_o} v^2 - V_{\text{eff}}^{(i)}(r), \tag{39}$$

$$p^{(i)}(r, v) = \frac{m_2^2(r)}{m_o} v, \tag{40}$$

and

$$H^{(i)}(r, p) = \frac{m_o p^2}{2m_2^2(r)} + V_{\text{eff}}^{(i)}(r)\ , \tag{41}$$

where $i = in$ for the incoming case, and $i = out$ for the outgoing case. As an example of illustration of this model, let us use the following parameters to estimate the dependence of several physical quantities with respect the parameter α,

$$\begin{aligned} &G = 6.67 \times 10^{-11}\ \text{m}^3/\text{Kg sec}\ ; \quad m_{oo} = 10^6\ \text{Kg}\ ; \quad m_f = 0.1\ m_{oo}\ ; \\ &P_\theta = 10^{17}\ \text{Kg m}^2/\text{sec}\ ; \quad \text{and} \quad K = -8 \times 10^{23}\ \text{Joules} \end{aligned} \tag{42}$$

Fig. 1 shows the curves of $V_{\text{eff}}(r)$ for several values of α (incoming case). As one can see from this figure, the location of the minimum does not change, but the minimum value of V_{eff} tends to disappear as α goes to zero. Also for the incoming

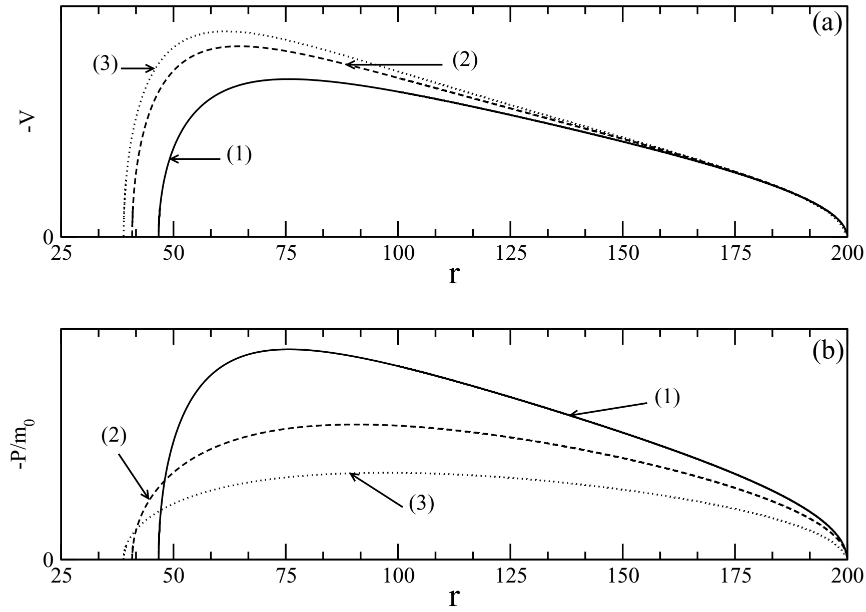

Fig. 3. (a): Trajectories in the plane (r, v); (b): Trajectories in the plane (r, p). $\alpha = 1$ (1), $\alpha = 0.01$ (2), and $\alpha = 0.005$ (3).

case, Fig. 2 shows how the minimum distance of approximation of the two bodies, r_1, and maximum distance, r_2, behave as a function of the parameter α. As one can guess, the following limit is satisfied $\lim_{\alpha\to 0} r_1 = \lim_{\alpha\to 0} r_2 = r^*$ which will become a inflexion point for V_{eff}. Fig. 3 shows the velocity (v) and normalized linear momentum (p/m_o) as a function of r for several values of α and for the incoming case. All the trajectories start at $r_2 = 200$ and finish at $r_1(\alpha)$. One can see the difference of the trajectories in (a) with respect to (b) due to position dependence of the momentum, relation (40).

5. CONCLUSIONS

The Lagrangian, Hamiltonian and a constant of motion of the gravitational attraction of two bodies when one of them has variable mass were given. One found that the relative and center of mass coordinates are coupled due to this mass variation. However, chosen the reference system in the much more massive body, it was possible to reduce the system to 1-D problem. Then, the constant of motion, Lagrangian and Hamiltonian were obtained. One main feature of these quantities was the appearance of an effective potential, which is reduced (when $\dot{m}_2 = 0$) to the usual gravitational effective potential of two bodies with fixed masses. Other feature was the distance dependence of the generalized linear momentum. A model for comet-mass-variation was given which depends on the parameter α. A study was made of the dependence with respect to α of V_{eff}, minimum and maximum distance between the two bodies, and the trajectories in the spaces (r, v) and (r, p). Of course, the problem of the interaction comet-star with the variation of mass deserves more complete analysis. The intention here with this example was to show explicitly the form of the constant of motion, Lagrangian, and Hamiltonian and to point out the different trajectories behavior in the spaces (r, v) and (r, p) arising from the constant of motion and Hamiltonian.

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